Year 13 Mathematics

IAS 3.15

Systems of Equations

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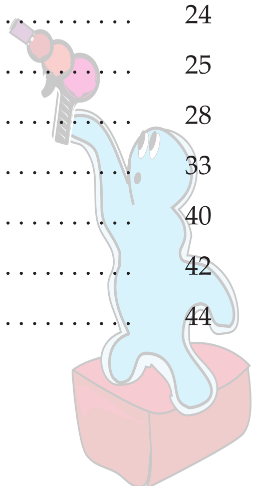

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NCEA 3 External Achievement Standard 3.15 – Apply Systems of Simultaneous Equations in Solving Problems.

This achievement standard involves applying systems of simultaneous equations in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply systems of simultaneous equations in solving problems.	• Apply systems of simultaneous equations, using relational thinking, in solving problems.	• Apply systems of simultaneous equations, using extended abstract thinking, in solving problems.

- This achievement standard is derived from Level 8 of The New Zealand Curriculum and is related to the achievement objective:
  - Form and use systems of simultaneous equations, including three linear equations in three variables, and interpret the solutions in context.
- Apply systems of simultaneous equations in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of concepts and terms
  - communicating using appropriate representations.
- Relational thinking involves one or more of:
  - selecting and carrying out a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts
  - forming and using a model;

  and also relating findings to a context, or communicating thinking using appropriate mathematical statements.
- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate or solve a problem
  - identifying relevant concepts in context
  - developing a chain of logical reasoning, or proof
  - forming a generalisation;

  and also using correct mathematical statements, or communicating mathematical insight.
- Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.
- Methods include a selection from those related to:
  - forming systems of simultaneous equations
  - solving systems of simultaneous equations
  - the nature of solutions to systems.

2 x 2 Simultaneous Equations

2 x 2 Simultaneous Equations

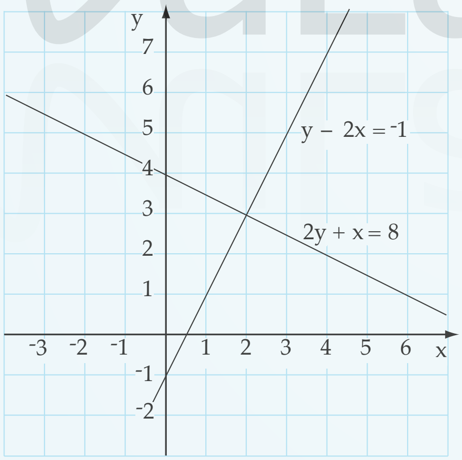
Simultaneous equations are sets of equations with more than one variable, and each equation has a solution of the same value for each variable. The equation $y = 2x + 3$ has many possible solutions including when $x = 4$ and $y = 11$. The equation $y = 15 - x$ also has many possible solutions, one of which is $x = 4$ and $y = 11$. Because the same values solve each of the equations we say they are simultaneously true for these equations.

We start our study with 2 x 2 simultaneous equations (although at level 3 NCEA we work with 3 x 3 simultaneous equations) because the concepts are the same and it is easier to understand these concepts with 2 x 2 simultaneous equations. 2 x 2 simultaneous equations are a pair of equations with up to two variables, hence two of two.

If we represent these two equations as straight lines on a Cartesian plane then graphically the solution is the single point through which the two lines pass. An example could be

$$2y + x = 8$$
$$y - 2x = {}^{-}1$$

If we plot these two lines we see they intersect at $x = 2$ and $y = 3$, the simultaneous solution to both equations.



There are a number of different methods for solving simultaneous equations. These include substitution, elimination, graphs, graphics calculator and matrices. From the graph it is easy to see the solution (or intersection), but the graphical approach has limited application as we cannot draw graphs in 3 dimensions (3 unknowns), which would be required to solve a system of 3 x 3 simultaneous equations.

In this booklet, the elimination, substitution and the graphics calculator methods are presented.

As a graphics calculator can solve linear – linear simultaneous equations you may think using algebra is superfluous.

It is true that most Achievement problems can be done using technology (graphics calculator or graphing programme on your device), but the Achievement Standard requires comment on 'the nature of solutions to systems of equations' and this will require an understanding and appreciation of the algebra involved in solving the simultaneous equations.

If you are only able to use the graphics calculator then you are limiting your understanding to Achievement level.

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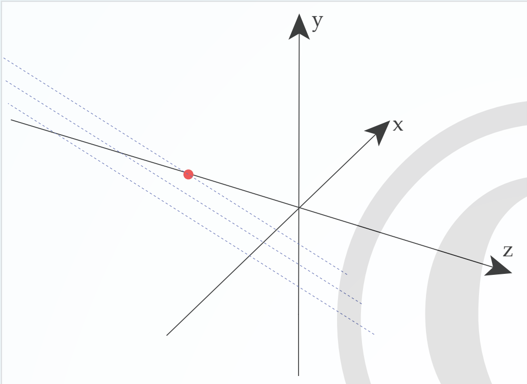
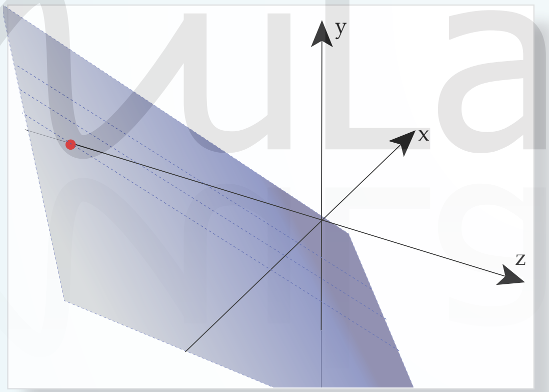
ax + by + cz = constant cont...

We could continue plotting points and extending them into lines where we know the z coordinate. For example, if we set $z = {}^{-}2$ we can draw the line $x + y = {}^{-}2$, where $z = {}^{-}2$.

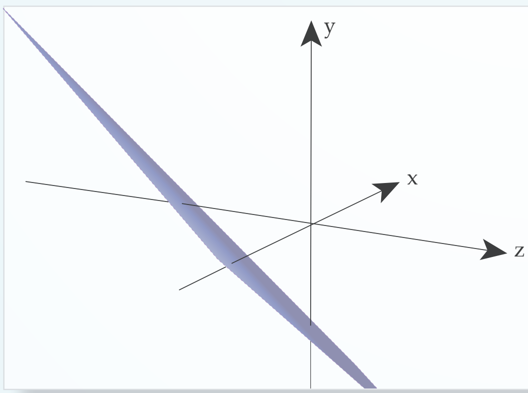


Remember these lines are in 3-dimensions.

Now we draw a plane through all the lines that are solutions to the original equation $x + y + z = {}^{-}4$. All the lines we have drawn up to now are on the surface of this plane we have constructed. The x, y and z co-ordinates of every point on this plane add to $^{-}4$.



If we twist the axes slightly we can look along the plane of points that add to $^{-}4$.



You can use technology to draw a three-dimensional plane and move it around.

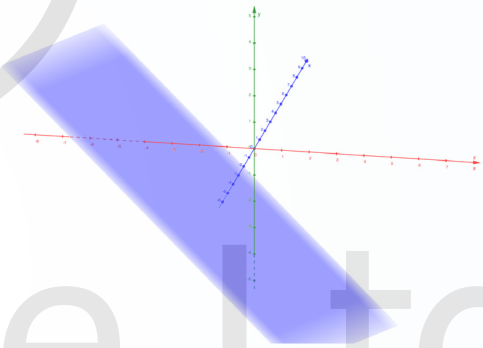
An example of a programme that is free and is available on any device is GeoGebra which is online but also has apps in the Google Play Store and iTunes App Store.

These instructions are for the online version but all versions should be similar.

Go to the web site GeoGebra.org and select GeoGebra 3D Graphing Calculator.

Prior to starting select the settings cog ⚙ and turnoff 'Show Plane' as by default it shows the x – y plane. Enter an equation (e.g. $x + y + z = -4$). You can move the plane around and inspect it from different angles.

You can change its colour by selecting the three vertical dots next to the equation and selecting 'Settings' then 'Color'.

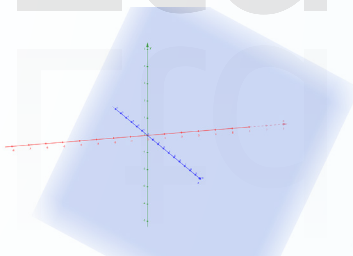


If we use GeoGebra to graph three planes:

Starting with

$x + 2y + z = 4.5 \quad (1)$

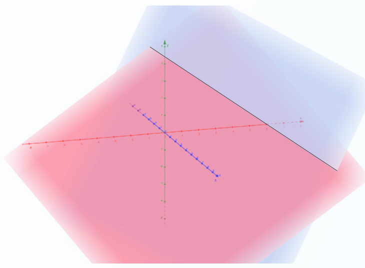
we get



If we draw

$2x - y - 2z = 12 \quad (2)$

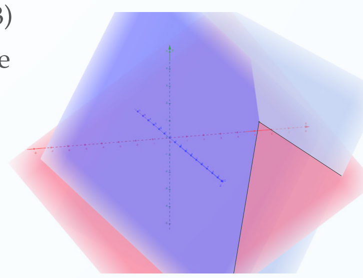
A second plane it intersects the first along a line.

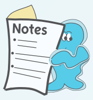


Adding a third plane

$^{-}2x + 3y + 3z = {}^{-}12 \quad (3)$

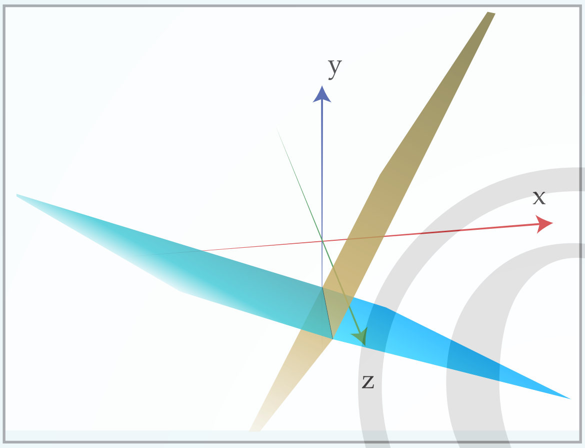
the intersection of three planes is usually a single point.



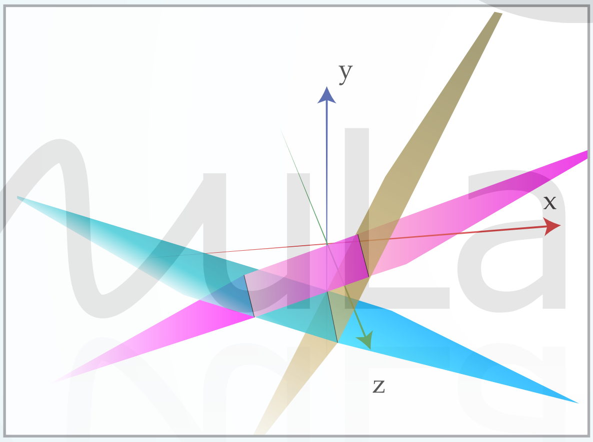


Inconsistency of Simultaneous Equations in Three-dimensions

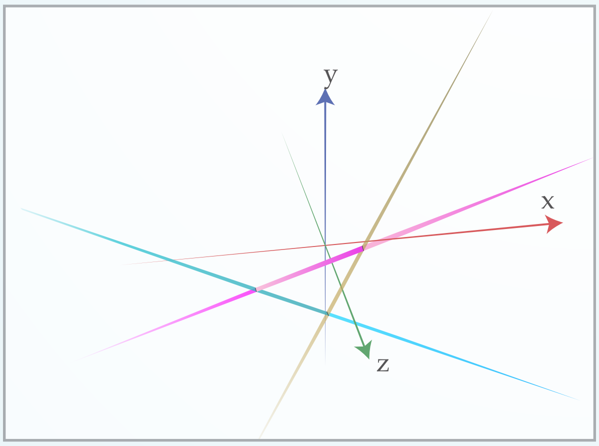
The intersection of two planes in three-dimensional space is a straight line.



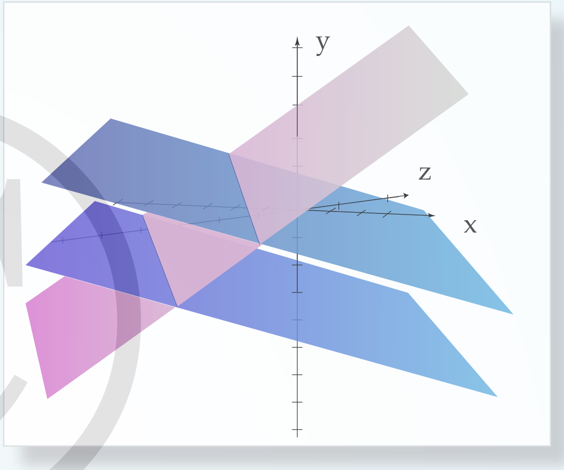
If the third plane runs parallel to this line of intersection there is no solution and the equations are inconsistent.



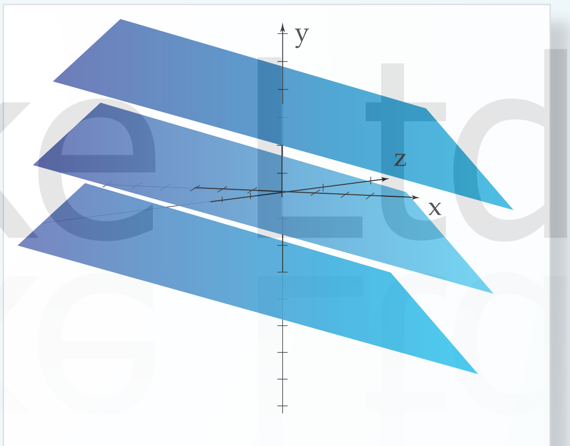
To show this we will twist our axes so we are almost looking along the line of intersection of all three planes. We can see that no matter how large the planes are, the three planes cannot meet at one point as these lines of intersection are parallel to each other.



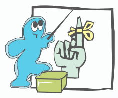
Other situations that give inconsistent results are when two planes are parallel. When we have **one pair of parallel planes** the third plane will cut both and these cut lines are parallel to each other.



Or when all three planes are parallel. With three parallel lines there is no intersection.



If you are attempting to solve any system of simultaneous equations and you get a contradictory statement such as $0 = {}^{-}28$, then they are inconsistent and cannot be solved.



Inconsistent Simultaneous Equations

With any system of simultaneous equations if you get any statement that can never be true there is no solution to the simultaneous equations.

To demonstrate this you will need to use algebra and not a graphics calculator.

Example

Demonstrate that the following system of equations has no unique answer. Find some of the answers.

$$\begin{array}{rcll} x + 2y + z &=& {}^-9 & (1) \\ x + y - 4z &=& 33 & (2) \\ {}^-2x - 3y + 3z &=& {}^-24 & (3) \end{array}$$

We multiply (2) by $^-1$ so the coefficients of x are the same but of opposite signs as equation (1).

$$\begin{array}{lrcll} {}^-1 \times (2) & {}^-x - y + 4z &=& {}^-33 & (4) \\ (1) + (4) & y + 5z &=& {}^-42 & (5) \end{array}$$

Now we multiply (1) by 2 and add to (3).

$$\begin{array}{lrcll} 2 \times (1) & 2x + 4y + 2z &=& {}^-18 & (6) \\ (6) + (3) & y + 5z &=& {}^-42 & (7) \end{array}$$

Combining equation (5) and equation (7) will eliminate both y and z and give a statement that is always true such as

$$0 = 0$$

This always true statement means there is no unique solution. The line of intersection of the first two planes lies on the third plane.

Now we assume that one solution for z is t.

$$\begin{array}{lrcll} \text{Assume} & z &=& t & \\ \text{Therefore} & x + 2y + t &=& {}^-9 & (1) \\ & x + 2y &=& {}^-9 - t & \\ & x + y - 4t &=& 33 & (2) \\ & x + y &=& 33 + 4t & \\ {}^-1 \times (1) & -x - 2y &=& 9 + t & \\ & x + y &=& 33 + 4t & \\ \text{Adding} & {}^-y &=& 42 + 5t & \\ & y &=& {}^-42 - 5t & \\ \text{Substituting} & x &=& 75 + 9t & \end{array}$$

When $t = 0$, $z = 0$, $y = {}^-42$ and $x = 75$

and when $t = {}^-8$, $z = {}^-8$, $y = {}^-2$ and $x = 3$

Example

Find the value of k which will mean that these simultaneous equation do NOT have a unique answer.

$$\begin{array}{rcll} 3x + 2y - z &=& 3 & (1) \\ 2x - 3y + 2z &=& 21 & (2) \\ {}^-7x - 9y + 5z &=& k & (3) \end{array}$$

Eliminating y we multiply (1) by 3 and (2) by 2.

$$\begin{array}{lrcll} 3 \times (1) & 9x + 6y - 3z &=& 9 & (4) \\ 2 \times (2) & 4x - 6y + 4z &=& 42 & (5) \\ (4) + (5) & 13x + z &=& 51 & (6) \end{array}$$

Now we multiply (2) by $^-3$ and add to (3).

$$\begin{array}{lrcll} {}^-3 \times (2) & {}^-6x + 9y - 6z &=& {}^-63 & (7) \\ (7) + (3) & {}^-13x - z &=& {}^-63 + k & (8) \end{array}$$

We recognise that (6) and (8) are parallel lines

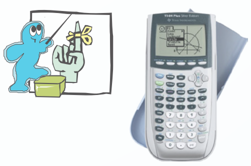
$$\begin{array}{rcll} 13x + z &=& 51 & (6) \\ {}^-13x - z &=& {}^-63 + k & (8) \end{array}$$

so we will either have inconsistent solutions or no unique solutions. Adding we get

$$\begin{array}{lrcl} (6) + (8) & 0 &=& {}^-12 + k \end{array}$$

If $k = 12$ then we do not have a unique solution.

If $k \neq 12$ then we have no solution at all.



Your TI-84 Plus calculator when using PLYSMLT2 will work out the system of possible answers as per the example on the left.

Your Casio 9750GII will give you a Ma ERROR but not find the equations of possible answers.

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Merit – Express each problem as a set of simultaneous equations and then solve the problem.

61. The local chess club runs a championship ladder where they allocate points for players who finish first, second or third in club competitions. At the end of the year three players were tied on 80 points. Logan has 8 wins, 5 seconds and 2 thirds. Shaun has 6 wins, 8 seconds and 3 thirds. Jerome has 4 wins, 10 seconds and 6 thirds.

How many points did the club allocate for first, second and third?

The club's solution to this tied result is to give one more point to each place, first, second and third. Who will now be the winner?

62. The electricity supply company charges a daily connection charge and different rates for night and day. Accounts for three months were

Month 1, days 28, night units 171 and day units 231 giving a total cost $91.77

Month 2, days 31, night units 196 and day units 212 giving a total cost $93.32

Month 3, days 25, night units 145 and day units 286 giving a total cost $97.35

What is the daily charge, night rate and day rate?

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Answers

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1. $x = 2$ and $y = {}^{-}3$

2. $x = 17$ and $y = 11$

Page 6

3. $x = {}^{-}2$ and $y = {}^{-}3$

4. $g = {}^{-}14$ and $h = 11.5$

5. $x = 2.20$ and $y = {}^{-}1.52$ (2 dp)

6. $x = 1.65$ and $y = 0.55$

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7. $x = {}^{-}1$ and $y = 1$

8. $x = 2.75$ and $y = 3.125$

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9. $x = 4.6$ and $y = 8.56$

10. $x = 2.26$ and $y = 0.687$ (3 sf)

11. $x = 163$ and $y = 435$

12. $x = 7.5$ and $y = 13.5$

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13. $x = 1.76$ and $y = 5.71$ (2 dp)

14. $x = 0.86$ and $y = 1.79$ (2 dp)

15. $(2k, 0)$

16. $(4, 2k + 4)$

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17. $x = 5$ and $y = {}^{-}3$

18. $x = {}^{-}0.5$ and $y = 0.25$

19. $x = 2$ and $y = {}^{-}11$

20. $x = 0.2$ and $y = {}^{-}0.15$

21. $x = {}^{-}6$ and $y = 0$

22. $x = 0.414$ and $y = 1.977$ (3 dp)

23. $x = 0$ and $y = 5$

24. $x = {}^{-}2$ and $y = {}^{-}2$

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25. $x = {}^{-}3$, $y = 4$ and $z = 2.5$

26. $x = 4$, $y = {}^{-}3$ and $z = {}^{-}2.5$

27. $x = 4$, $y = {}^{-}1$ and $z = 7$

28. $x = 2$, $y = 0$ and $z = {}^{-}3$

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29. $x = 3$, $y = {}^{-}2$ and $z = {}^{-}1$

30. $f = 11$, $g = {}^{-}4$ and $h = 3.5$

31. $a = 4.2$, $b = 1.6$ and $c = 0.8$

32. $p = 8$, $q = 3$ and $r = 1$

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33. $x = 2$, $y = 1$ and $z = {}^{-}3$

34. $x = \frac{1}{2}$, $y = \frac{{}^{-}1}{3}$ and $z = \frac{5}{6}$

35. $x = 2$, $y = 3$ and $z = {}^{-}2$

36. $x = 2$, $y = {}^{-}0.5$ and $z = 6$

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37. $x = 1$, $y = 3$ and $z = {}^{-}2$

38. $x = 3.5$, $y = 4$ and $z = {}^{-}2$

39. $x = 1.5$, $y = {}^{-}2.5$ and $z = 0$

40. $x = 1$, $y = {}^{-}2$ and $z = {}^{-}3$

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41. $x = 2$, $y = {}^{-}5$ and $z = 4$

42. $x = 12$, $y = {}^{-}8$ and $z = 20$

43. $x = 6.23$, $y = {}^{-}1.81$ and $z = {}^{-}2.57$ (3 sf)

44. $x = 1.06$, $y = 1.46$ and $z = 0.962$ (3 sf)

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45. Attempting to eliminate any variable gives a contradictory statement such as
$0 = 22$
So equations are inconsistent.

46. Attempting to eliminate any variable gives a contradictory statement such as
$0 = {}^{-}3$
So equations are inconsistent.

47. Attempting to eliminate any variable gives a contradictory statement such as
$0 = 20$
So equations are inconsistent.

48. Attempting to eliminate any variable gives a contradictory statement such as
$0 = {}^{-}3$
So equations are inconsistent.

49. Unique solution
$x = 2$, $y = 3$ and $z = 4$.

50. Contradictory statement results so equations are inconsistent.

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51. A true statement results so no unique solution.

52. A true statement results so no unique solution.

53. A true statement results so no unique solution.

54. A true statement results so no unique solution.

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55. Unique solution
$x = 4$, $y = {}^{-}2$ and $z = 1$.

56. A true statement results so no unique solution.

57. $a = {}^{-}7$. Results in a contradictory statement results so equations are then inconsistent.

58. $c = 8$. Results in a statement that is always true so this would mean no unique solution.

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59. Multiple answers possible including $x + y + z = 3$. The point $(2, 5, {}^{-}4)$ must solve the equation and it must not solve with the point $(3, {}^{-}3, 1)$ [which solves the first two but for a unique answer should not solve the generated answer].

60.
$$8y - 9z = 10 - 2k$$
and $8y - 9z = 6$

gives $10 - 2k = 6$

$$k = 2$$

When $k = 2$ an answer is possible (consistent) but as this reduces the simultaneous equations to a result that is always true, the answer is not unique.

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61.
$8F + 5S + 2T = 80$
$6F + 8S + 3T = 80$
$4F + 10S + 6T = 80$
$F = 7$, $S = 4$ and $T = 2$.

Change means Jerome wins.

62.
$28C + 171N + 231D = 9177$
$31C + 196N + 212D = 9332$
$25C + 145N + 286D = 9735$
Connection = 120¢ /d
Night rate = 7¢ /u
Day rate = 20¢ /u